Logistic approximation to the logistic-normal integral

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http://threeplusone.com/logistic-normal

Gavin E. Crooks

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The logistic-normal integral

$$g(x;\alpha,\sigma) = \int_{-\infty}^{+\infty} f(y;\alpha)\,\mathcal{N}(y;x,\sigma)\,dy\,, \tag{1}$$

is a convolution of a logistic function

$$f(x;\alpha) = \frac{1}{1+e^{-x/\alpha}} = \frac{1}{2} + \frac{1}{2}\tanh\frac{x/\alpha}{2}, \tag{2}$$

with a normal distribution

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{3}$$

This Weierstrass transform of the logistic function has important applications to logistic regression with normally distributed measurement error [1–3]

Here we demonstrate the integral is approximately a reparameterized logistic function:

$$g(x;\alpha,\sigma) \approx f(x;\gamma\alpha)\,, \quad \gamma = \sqrt{1+\frac{\pi\sigma^2}{8\alpha^2}} \tag{4}$$

The absolute maximum error is less than 0.02.

We proceed by noting that a logistic function can be closely approximated by the error function

$$f(x,\alpha) \approx \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\pi}}{4\alpha}x\right). \tag{5}$$

The scaling factors are fixed by requiring equality of the derivative at the origin, since, for our purposes, it is more important to minimize the errors around the origin than elsewhere.

The convolution of an error function and a normal distribution is another error function.

$$\int_{-\infty}^{+\infty}\left[\frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\pi}}{4\alpha}y\right)\right]\frac{e^{-\frac{(y-x)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy = \frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\pi}}{4\alpha\gamma}x\right)$$

By applying approximation (5) again, but in reverse, we obtain the desired result, Eq. 6.

This approximation to the logistic-normal integral is simple, analytic, reasonable accurate, and exact in the small noise limit ($\sigma \to 0$). A variety of more complex approximations to the logistic-normal integral have been investigated [1, 2], and if more precession is necessary recent advances allow for the rapid numerical evaluation of the integral [4].

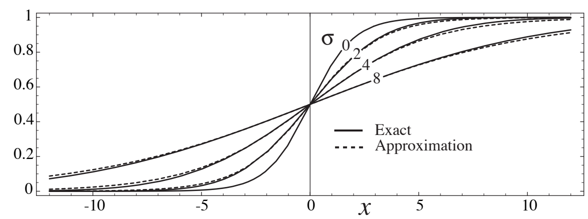

Figure 1: The approximation of the sigmoidal function $g(x;\alpha,\sigma)$ [Eq. (1)] by the logistic function $f(x;\gamma) = 1/(1+\exp(-x/\gamma))$, where $\gamma = \sqrt{1+\pi\sigma^2/8}$ [Eq. (6)]. The absolute difference between the functions is always less than 0.02.

It is interesting to note that the same basic approach can be applied to the Weierstrass transform of any sigmoid function. Approximate the sigmoid with an error function by equating derivatives at zero, perform the transform, then reverse the approximation to recover a rescaled sigmoid in the original functional form.

$$g(x;\alpha,\sigma) \approx f(x;\gamma\alpha)\,,\ \gamma = \sqrt{1+\frac{2\pi\sigma^2}{r}}\,,\ r = \frac{d}{dx}f(x;\alpha)\Big|_0 \tag{6}$$

The only parameter of the approximation, r is the derivative of the sigmoidal function evaluated at zero.

A more convoluted derivation of this logistic approximation was originally published as an appendix to Maragakis et al[3].

References

[1] E. Crouch and D. Spiegelman. The evaluation of integrals of the form $\int_{-\infty}^{+\infty} f(t)\exp(-t^2)dt$: Application to logistic-normal models. *J. Amer. Statist. Assoc.*, 85(410):464–469, 1990.

[2] E. Demidenko. *Mixed Models*. Wiley-Interscience, 2004.

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nonequilibrium work data in the presence of instrument noise. *J. Chem. Phys.*, 129:024102, Jul 2008. doi:10.1063/1.2937892.

[4] D. Pirjol. The logistic-normal integral and its generalizations. *J. Comp. Appl. Math.*, 237(1):460–469, 2013. doi:10.1016/j.cam.2012.06.016.

Versions: v1 (2007): More convoluted derivation, published as an appendix to Maragakis (2008) [3]. v2 (2009): Simplified derivation. v3 (2012): Minor changes to text and formatting. v4 (2013) Add generalization to sigmoid-normal integral. Add citation to numerical evaluation of integral.