Approximate convolution of a logistic function with a Gaussian distribution.

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We are interested in the function

$$g(x;\alpha,\sigma) = \int_{-\infty}^{+\infty} f(x+\epsilon;\alpha)\,\mathcal{N}(\epsilon;0,\sigma)d\epsilon\,, \qquad (1)$$

the convolution of a logistic (or Fermi) function

$$f(x;\alpha) = \frac{1}{1+\mathrm{e}^{-x/\alpha}} = \frac{1}{2} + \frac{1}{2}\tanh\frac{x/\alpha}{2}, \qquad (2)$$

with a Gaussian (or normal) distribution with zero mean and standard deviation σ:

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \qquad (3)$$

The function $g(x;\alpha,\sigma)$ does not have a simple, closed form. However, as is illustrated in the figure, it can be reasonably approximated by a reparameterized logistic function:

$$g(x;\alpha,\sigma) \approx f(x;\gamma), \qquad (4)$$

where γ is a function of α and σ. We fix γ by requiring equality of the derivative at the origin, since, for our purposes, it is more important to minimize the errors around the origin than elsewhere. The value of $g(x;\alpha,\sigma)$ at the origin is $1/2$, the same as $f(0;\gamma)$. Note that

$$\left.\frac{d}{dx}f(x;\gamma)\right|_{x=0} = \left.\frac{1}{2\gamma+2\gamma\cosh\left(x/\gamma\right)}\right|_{x=0} = \frac{1}{4\gamma}, \qquad (5)$$

and therefore

$$\begin{aligned}
\gamma^{-1} &= 4\left.\frac{d}{dx}g(x;\alpha,\sigma)\right|_{x=0} \\
&= 4\int_{-\infty}^{+\infty}\left(\left.\frac{d}{dx}f(x+\epsilon;\alpha)\right|_{x=0}\right)\mathcal{N}(\epsilon;\sigma)d\epsilon \\
&= 4\int_{-\infty}^{+\infty}\left(\frac{1}{2\alpha+2\alpha\cosh\epsilon/\alpha}\right)\mathcal{N}(\epsilon;\sigma)d\epsilon. \qquad (6)
\end{aligned}$$

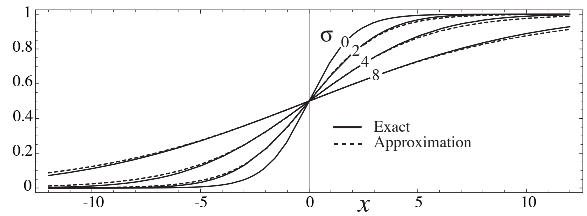

FIG. 1: The approximation of the sigmoidal function $g(x;\alpha,\sigma)$ [Eq. (1)] by the logistic function $f(x;\gamma) = 1/(1+\exp(-x/\gamma))$, where $\gamma = \sqrt{1+\pi\sigma^2/8}$ [Eq. (7)]. The absolute difference between the functions is always less than 0.02.

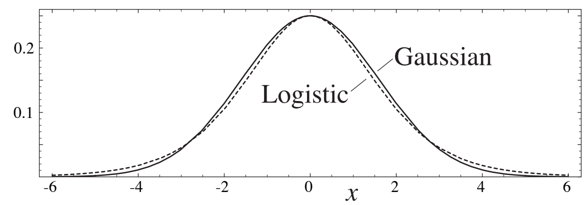

FIG. 2: The approximation of the standard logistic distribution by the Gaussian distribution with zero mean and standard deviation $\sqrt{8/\pi}$.

The expression inside the bracket is a logistic distribution, which is closely approximated by the Gaussian distribution $\mathcal{N}(\epsilon;0,\alpha\sqrt{8/\pi})$ (See Fig. 2). These parameters ensure that the two distributions agree exactly at the origin. Therefore, our problem reduces to a straightforward Gaussian integral:

$$\begin{aligned}
\gamma^{-1} &\approx 4\int_{-\infty}^{+\infty}\mathcal{N}(\epsilon;0,\alpha\sqrt{\frac{8}{\pi}})\,\mathcal{N}(\epsilon;0,\sigma)d\epsilon \\
\gamma &= \sqrt{1+\frac{\pi}{8\alpha^2}\sigma^2}. \qquad (7)
\end{aligned}$$

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[1] P. Maragakis, F. Ritort, M. Karplus, C. Bustamante, and G. E. Crooks (2007).