Gates, States, and Circuits:

Notes on the circuit model of quantum compuation

Tech. Note 014v1 http://threeplusone.com/gates

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Contents

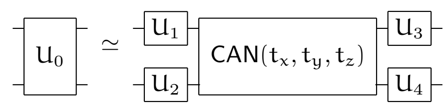
1 The canonical gate

The canonical gate is a 3-parameter quantum logic gate that acts on two qubits.

$$\begin{aligned}&\text{CAN}(t_x, t_y, t_z)\\&\quad = \exp\Big(-i\frac{\pi}{2}(t_x X \otimes X + t_y Y \otimes Y + t_z Z \otimes Z)\Big)\end{aligned} \quad (1)$$

Here, $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the 1-qubit Pauli matrices. Note that other choices for the prefactor in the exponential are also common in the literature.

The canonical gate is, in a sense, the elementary 2-qubit gate, since any other 2-qubit gate can be decomposed into a canonical gate, and local 1-qubit interactions [2, 3, 4, 5].



(2)

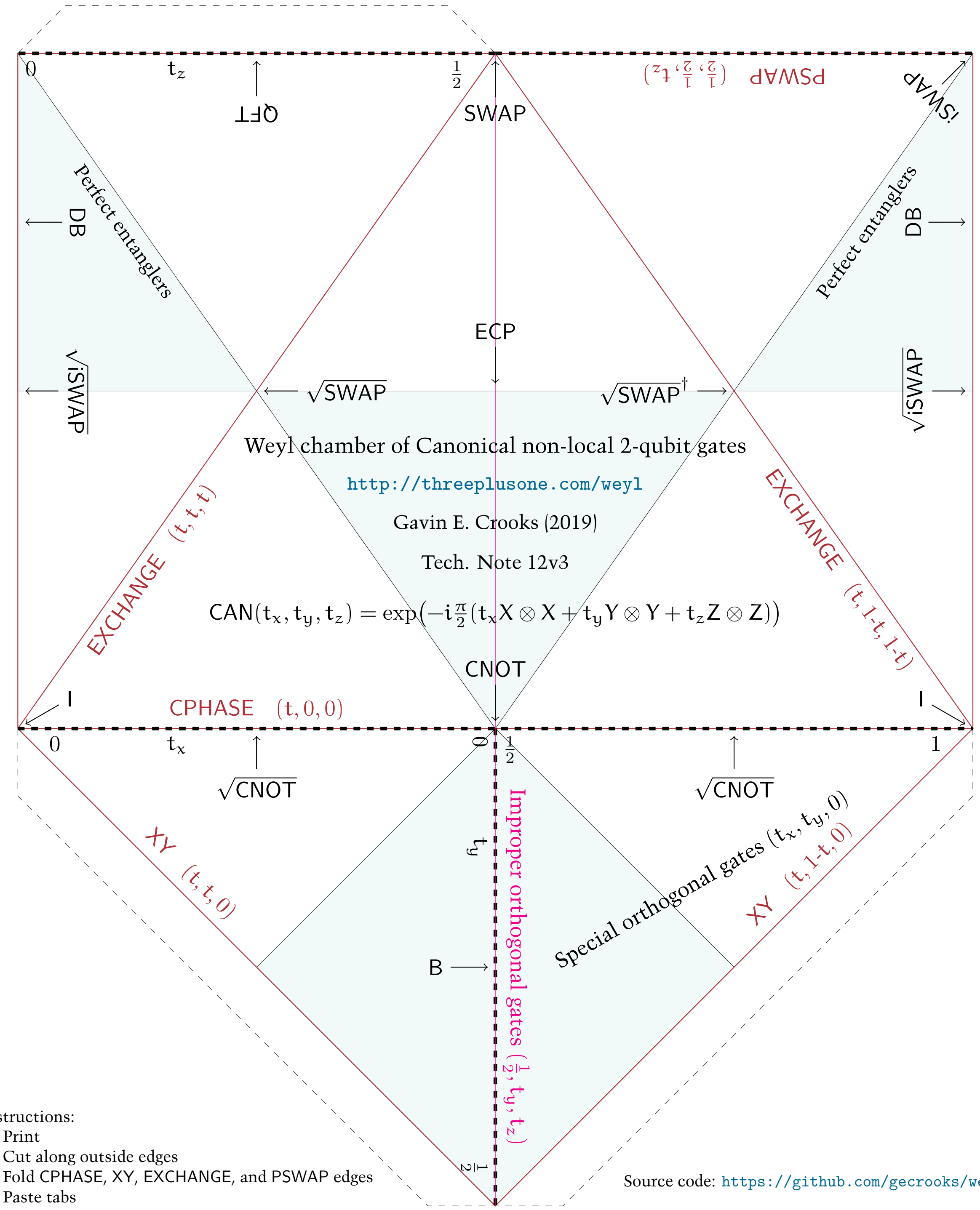
Here we use '$\simeq$' to indicate that two gates have the same unitary operator up to a global (and generally irrelevant) phase factor.

The canonical gate is periodic in each parameters with period 4, or period 2 if we neglect a -1 global phase factor. Thus we can constrain each parameter to the range $[-1, 1)$. Since $X \otimes X$, $Y \otimes Y$, and $Z \otimes Z$ all commute, the parameter space has the topology of a 3-torus.

However, the canonical coordinates of any given 2-qubit gate are not unique since we have considerable freedom in the prepended and postpended local gates. To remove these symmetries we can constraint the canonical parameters to a "Weyl chamber" [1, 1].

$$(\tfrac{1}{2} \geqslant t_x \geqslant t_y \geqslant t_z \geqslant 0) \cup (\tfrac{1}{2} \geqslant (1 - t_x) \geqslant t_y \geqslant t_z > 0) \quad (3)$$

This Weyl chamber forms a trirectangular tetrahedron. All gates in the Weyl chamber are locally inequivalent (They cannot be obtained from each other via local 1-qubit gates). The net of the Weyl chamber is illustrated in Fig. 1, and the coordinates of many common 2-qubit gates are listed in table 1. Code for performing a canonical-decomposition, and therefore of determining the Weyl coordinates, can be found in the decompositions subpackage of `QuantumFlow` [6].



Instructions:
(1) Print
(2) Cut along outside edges
(3) Fold CPHASE, XY, EXCHANGE, and PSWAP edges
(4) Paste tabs

Source code: https://github.com/gecrooks/weyl

Figure 1: The Weyl chamber of canonical non-local 2 qubit gates. (Print, cut, fold, and paste)

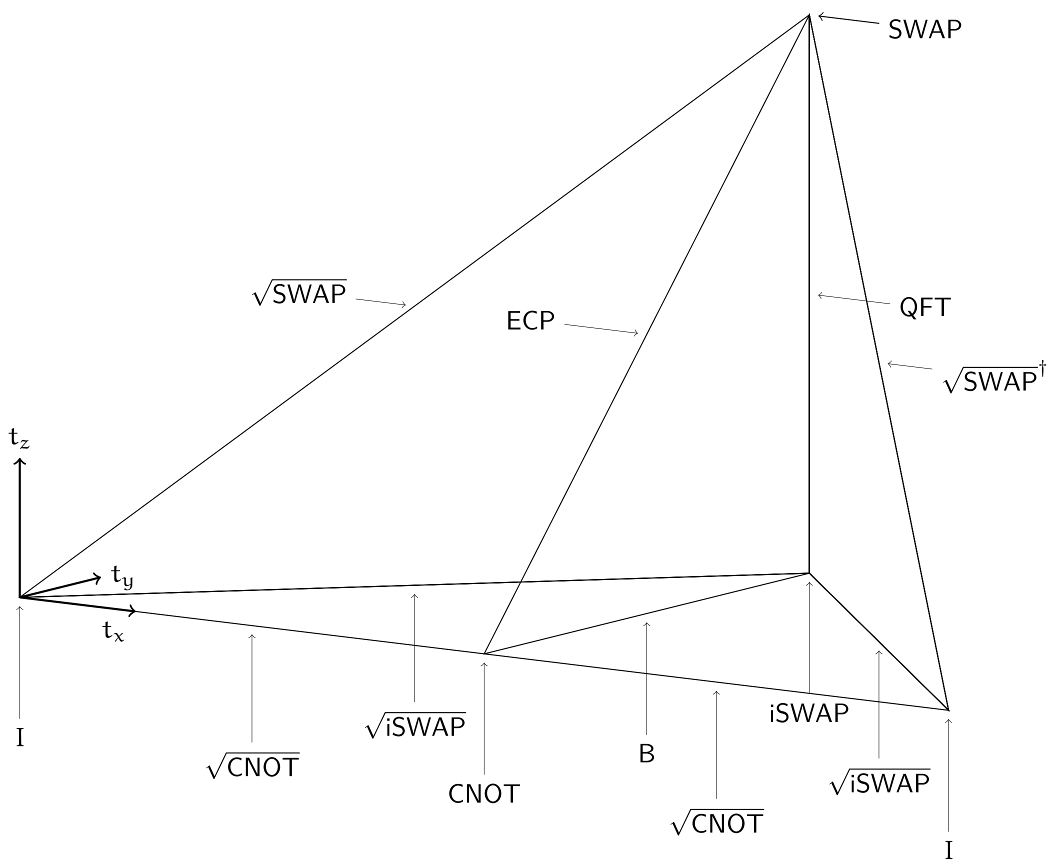


Figure 2: Location of principle 2-qubit gates in the Weyl chamber. All of these gates have coordinates of the form $\mathsf{CAN}(\frac{k}{4}, \frac{m}{4}, \frac{n}{4})$, for integer k, m, and n.

Table 1: Canonical coordinates of common 2-qubit gates

Gate	t_x	t_y	t_z	t'_x	t'_y	t'_z
	$\leqslant \frac{1}{2}$			$> \frac{1}{2}$		
I_2	0	0	0	1	0	0
CNOT / CZ	$\frac{1}{2}$	0	0			
iSWAP / DCNOT	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{1}{2}$	0
SWAP	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			
$\sqrt{\text{CNOT}}$	$\frac{1}{4}$	0	0	$\frac{3}{4}$	0	0
$\sqrt{\text{iSWAP}}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{3}{4}$	$\frac{1}{4}$	0
DB	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{5}{8}$	$\frac{3}{8}$	0
$\sqrt{\text{SWAP}}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$			
$\sqrt{\text{SWAP}}^\dagger$				$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{2}$	$\frac{1}{4}$	0			
ECP	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$			
QFT_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$			

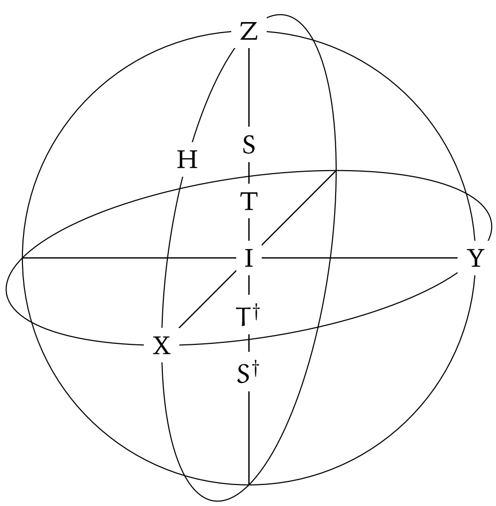


Figure 3: Sphere of 1-qubit gates

2 Principle 2-qubit gates

We use '≅' to indicate that two gates are locally equivalent, in that they can be mapped to one another by local, 1-qubit rotations.

2.1 Clifford gates

There are four unique 2-qubits gates in the Clifford group (up to local 1-qubit Cliffords): the identity, CNOT, iSWAP, and SWAP gates.

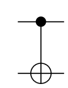
Identity gate

$$I_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} = \text{CAN}(0,0,0) \tag{4}$$

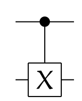
Controlled NOT (Controlled-X) gate

$$\text{CNOT} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \cong \text{CAN}(\tfrac{1}{2},0,0) \tag{5}$$

Commonly represented by the circuit diagrams

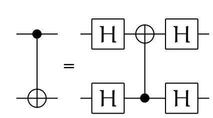


or



.

The CNOT gate is not symmetric between the two qubits. But we can switch control ● and target ⊕ with local Hadamard gates.



iSWAP-gate

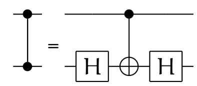
$$\text{iSWAP} = \begin{pmatrix} 1&0&0&0\\ 0&0&i&0\\ 0&i&0&0\\ 0&0&0&1 \end{pmatrix} \cong \text{CAN}(\tfrac{1}{2},\tfrac{1}{2},0) \tag{6}$$

SWAP-gate

$$\text{SWAP} = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix} \cong \text{CAN}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) \tag{7}$$

2.2 XX gates

Gates in the XX (or Ising) class have coordinates CAN$(t, 0, 0)$, which forms the one edge of the Weyl chamber. This includes the identity and CNOT gates.



XX (Ising) gate

$$
\begin{aligned}
\mathrm{XX}(t) &= e^{-i\frac{\pi}{2} X\otimes X} \\
&= \begin{pmatrix} \cos(\frac{\pi}{2}t) & 0 & 0 & -i\sin(\frac{\pi}{2}t) \\ 0 & \cos(\frac{\pi}{2}t) & -i\sin(\frac{\pi}{2}t) & 0 \\ 0 & -i\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) & 0 \\ -i\sin(\frac{\pi}{2}t) & 0 & 0 & \cos(\frac{\pi}{2}t) \end{pmatrix} \\
&= \mathrm{CAN}(t, 0, 0)
\end{aligned} \tag{8}
$$

YY gate

$$
\begin{aligned}
\mathrm{YY}(t) &= e^{-i\frac{\pi}{2} Y\otimes Y} \\
&= \begin{pmatrix} \cos(\frac{\pi}{2}t) & 0 & 0 & +i\sin(\frac{\pi}{2}t) \\ 0 & \cos(\frac{\pi}{2}t) & -i\sin(\frac{\pi}{2}t) & 0 \\ 0 & -i\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) & 0 \\ +i\sin(\frac{\pi}{2}t) & 0 & 0 & \cos(\frac{\pi}{2}t) \end{pmatrix} \\
&= \mathrm{CAN}(0, t, 0) \\
&\cong \mathrm{CAN}(t, 0, 0)
\end{aligned} \tag{9}
$$

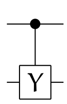
ZZ gate

$$
\begin{aligned}
\mathrm{ZZ}(t) &= e^{-i\frac{\pi}{2} Z\otimes Z} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\pi t} & 0 & 0 \\ 0 & 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \mathrm{CAN}(0, 0, t) \\
&\cong \mathrm{CAN}(t, 0, 0)
\end{aligned} \tag{10}
$$

Controlled-Y gate

$$
\begin{aligned}
\mathrm{CY} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} \\
&\cong \mathrm{CAN}(\tfrac{1}{2}, 0, 0)
\end{aligned} \tag{11}
$$

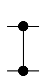
Commonly represented by the circuit diagram:


(12)

Controlled-Z gate

$$
\begin{aligned}
\mathrm{CZ} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&\cong \mathrm{CAN}(\tfrac{1}{2}, 0, 0)
\end{aligned} \tag{13}
$$

Commonly represented by the circuit diagrams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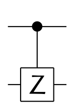
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Square root CNOT gate

$$
\begin{aligned}
\sqrt{\mathrm{CNOT}} &= \begin{pmatrix} \cos(\frac{\pi}{8}) & 0 & 0 & -i\sin(\frac{\pi}{8}) \\ 0 & \cos(\frac{\pi}{8}) & -i\sin(\frac{\pi}{8}) & 0 \\ 0 & -i\sin(\frac{\pi}{8}) & \cos(\frac{\pi}{8}) & 0 \\ -i\sin(\frac{\pi}{8}) & 0 & 0 & \cos(\frac{\pi}{8}) \end{pmatrix} \\
&\cong \mathrm{CAN}(\tfrac{1}{4}, 0, 0)
\end{aligned} \tag{14}
$$

2.3 XY gates

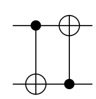
Gates in the XY class forms two edges of the Weyl chamber with coordinates CAN$(t, t, 0)$ (for $t \leqslant \frac{1}{2}$) and CAN$(t, 1-t, 0)$ (for $t > \frac{1}{2}$). This includes the identity and iSWAP gates.

XY-gate Also occasionally referred to as the piSWAP (or parametric iSWAP) gate.

$$
\begin{aligned}
\mathrm{XY}(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi t) & -i\sin(\pi t) & 0 \\ 0 & -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \mathrm{CAN}(t, t, 0) \\
&\cong \mathrm{CAN}(t, 1-t, 0)
\end{aligned} \tag{15}
$$

Double Controlled NOT (DCNOT) gate

$$
\begin{aligned}
\mathrm{DCNOT} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&\cong \mathrm{CAN}(\tfrac{1}{2}, \tfrac{1}{2}, 0)
\end{aligned} \tag{16}
$$

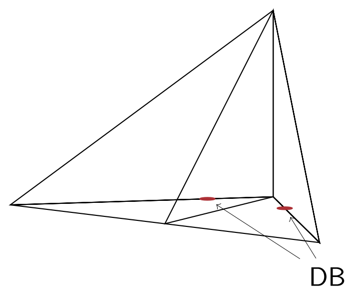


bSWAP (Bell-Rabi) gate [?]

$$
\begin{aligned}
\mathrm{bSWAP} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\
&= \mathrm{CAN}(\tfrac{1}{2}, -\tfrac{1}{2}, 0) \\
&\cong \mathrm{CAN}(\tfrac{1}{2}, \tfrac{1}{2}, 0)
\end{aligned} \tag{17}
$$

Dagwood Bumstead (DB) gate [7] Of all the gates in the XY class, the Dagwood Bumstead-gate makes the biggest sandwiches. [7, Fig. 4]

$$
\begin{aligned}
\mathsf{DB} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\frac{3\pi}{8}) & -i\sin(\frac{3\pi}{8}) & 0 \\ 0 & -\sin(\frac{3\pi}{8}) & \cos(\frac{3\pi}{8}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \mathsf{XY}(\tfrac{3}{8}) \\
&= \mathsf{CAN}(\tfrac{3}{8}, \tfrac{3}{8}, 0)
\end{aligned}
\tag{18}
$$

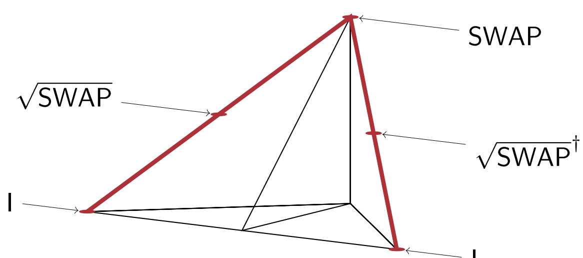


2.4 Exchange-interaction gates

Includes the identity and SWAP gates.

EXCH (XXX) gate

$$\mathsf{EXCH}(t) = \mathsf{CAN}(t, t, t) \tag{19}$$



Exchange gates

√SWAP-gate

$$\sqrt{\mathsf{SWAP}} = \mathsf{CAN}(\tfrac{1}{4}, \tfrac{t}{4}, \tfrac{t}{4}) \tag{20}$$

Inverse √SWAP-gate

$$\sqrt{\mathsf{SWAP}}^{\dagger} = \mathsf{CAN}(\tfrac{3}{4}, \tfrac{t}{4}, \tfrac{t}{4}) \tag{21}$$

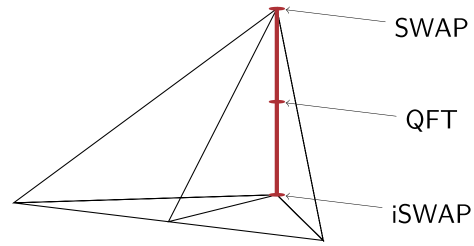
Because of the symmetry around $t_x = \frac{1}{2}$ on the base of the Weyl chamber, the CNOT and iSWAP gates only have one square root. But the SWAP has two unique square roots, which are inverses of each other.

2.5 Parametric SWAP gates

The class of parametric SWAP (PSWAP) gates forms the remaining edge of the Weyl chamber, connecting the SWAP and iSWAP gates.

pSWAP gate [1]

$$\mathsf{pSWAP}(t) \cong \mathsf{CAN}(\tfrac{1}{2}, \tfrac{1}{2}, t) \tag{22}$$



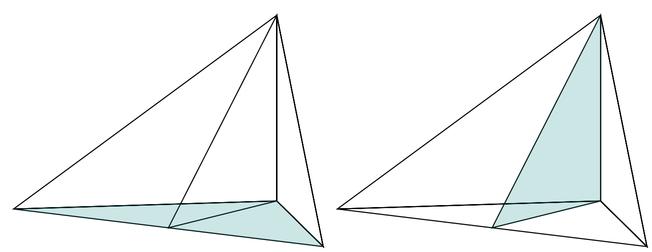
pSWAP gates

2-qubit quantum Fourier transform (QFT) [1]

$$
\begin{aligned}
\mathsf{QFT} &= \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \\
&\cong \mathsf{CAN}(\tfrac{1}{2}, \tfrac{1}{4}, 0)
\end{aligned}
\tag{23}
$$

2.6 Orthogonal gates

An orthogonal gate, in this context, is a gate that can be represented by an orthogonal matrix (up to local 1-qubit rotations.) The special orthogonal gates have determinant +1 and coordinates $\mathsf{CAN}(t_x, t_y, 0)$, which covers the bottom surface of the canonical Weyl chamber. The improper orthogonal gates have determinant −1 and coordinates $\mathsf{CAN}(\frac{1}{2}, t_y, t_z)$, which is a plane connecting the CNOT, iSWAP, and SWAP gates.



Special orthogonal gates   Improper orthogonal gates

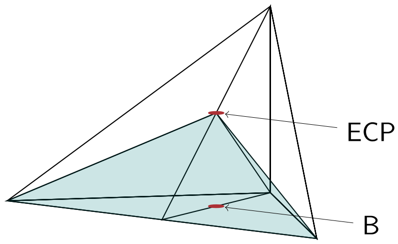
B (Berkeley) gate [1]

$$
\begin{aligned}
\mathsf{B} &= \begin{pmatrix} \cos(\frac{\pi}{8}) & 0 & 0 & i\sin(\frac{\pi}{8}) \\ 0 & \cos(\frac{3\pi}{8}) & i\sin(\frac{3\pi}{8}) & 0 \\ 0 & i\sin(\frac{3\pi}{8}) & \cos(\frac{3\pi}{8}) & 0 \\ i\sin(\frac{\pi}{8}) & 0 & 0 & \cos(\frac{\pi}{8}) \end{pmatrix} \\
&\cong \mathsf{CAN}(\tfrac{1}{2}, \tfrac{1}{4}, 0)
\end{aligned}
\tag{24}
$$

Notably two-B gates are enough to create any other 2-qubit gate. Unfortunately the B gate isn't a particularly natural gate on any current hardware.

ECP-gate [7]

$$\mathrm{ECP} = \mathrm{CAN}(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}) \tag{25}$$

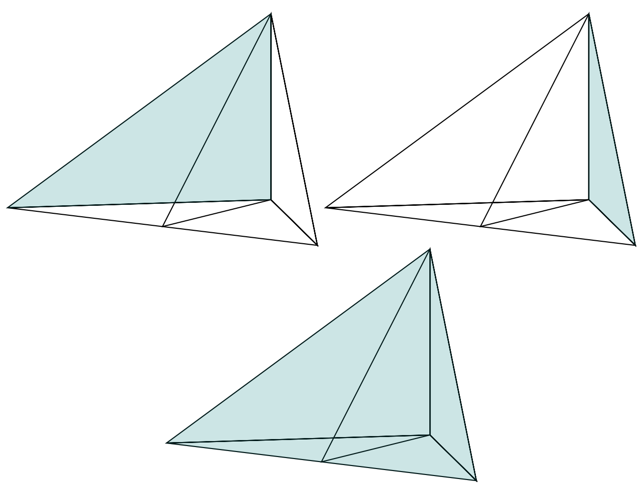


B and ECP gates, and ECP pyramid

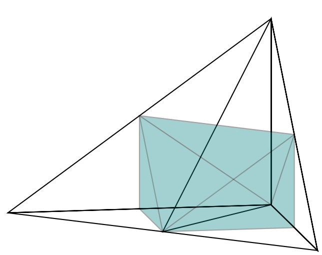
2.7 XXY gates

The remaining faces of the Weyl chamber are the XXY family. Thanks to the Weyl symmetries, this family covers all three faces that meet at the SWAP gate.

$$\mathrm{XXY}(t, \delta) = \mathrm{CAN}(t, t, \delta) \tag{26}$$



2.8 Perfect entanglers


Perfect entaglers

Acknowledgments Consideration of importance of the canonical gates and gate decompositions arose from many conversations with Eric C. Peterson.

References

[1] [citation needed]. (pages 1, 1, 6, 6, and 6).

[2] Jun Zhang, Jiri Vala, Shankar Sastry, and K. Birgitta Whaley. Geometric theory of nonlocal two-qubit operations. *Phys. Rev. A*, 67:042313 (2003). doi:10.1103/PhysRevA.67.042313. ArXiv:quant-ph/0209120. (page 1).

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[7] Eric C. Peterson, Gavin E. Crooks, and Robert S. Smith. Fixed-depth two-qubit circuits and the monodromy polytope. arXiv:1904.10541. (pages 5, 5, and 6).

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