1 The canonical gate

The canonical gate is a 3-parameter quantum logic gate that acts on two qubits.

\[
\text{CAN}(t_x, t_y, t_z) = \exp \left( -\frac{i \pi}{2} (t_x X \otimes X + t_y Y \otimes Y + t_z Z \otimes Z) \right) \tag{1}
\]

Here, \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), and \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) are the 1-qubit Pauli matrices. Note that other choices for the prefactor in the exponential are also common in the literature.

The canonical gate is, in a sense, the elementary 2-qubit gate, since any other 2-qubit gate can be decomposed into a canonical gate, and local 1-qubit interactions \([2, 3, 4, 5]\).

\[
\begin{array}{c}
\begin{array}{c}
\text{U}_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{U}_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{CAN}(t_x, t_y, t_z)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{U}_3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{U}_4
\end{array}
\end{array}
\end{array}
\tag{2}
\]

Here we use \( \simeq \) to indicate that two gates have the same unitary operator up to a global (and generally irrelevant) phase factor.

The canonical gate is periodic in each parameters with period 4, or period 2 if we neglect a \(-1\) global phase factor. Thus we can constrain each parameter to the range \([-1, 1]\). Since \( X \otimes X, Y \otimes Y, \) and \( Z \otimes Z \) all commute, the parameter space has the topology of a 3-torus.

However, the canonical coordinates of any given 2-qubit gate are not unique since we have considerable freedom in the prepended and postpended local gates. To remove these symmetries we can constrain the canonical parameters to a “Weyl chamber” \([1, 1]\).

\[
(\frac{1}{2} \geq t_x \geq t_y \geq t_z) \cup (\frac{1}{2} \geq 1-t_x \geq t_y \geq t_z > 0) \tag{3}
\]

This Weyl chamber forms a trirectangular tetrahedron. All gates in the Weyl chamber are locally inequivalent (they cannot be obtained from each other via local 1-qubit gates). The net of the Weyl chamber is illustrated in Fig. 1, and the coordinates of many common 2-qubit gates are listed in table 1. Code for performing a canonical-decomposition, and therefore of determining the Weyl coordinates, can be found in the decompositions subpackage of QuantumFlow \([6]\).
Perfect entanglers

Improper orthogonal gates $(t_x, t_y, t_z)$

Special orthogonal gates $(t_x, t_y, 0)$

Exchange $(t, t, t)$

CPHASE $(t, 0, 0)$

EXCHANGE $(t, t, t)$

PSWAP $(1, 2, t_z)$

SWAP

Weyl chamber of Canonical non-local 2-qubit gates

http://threeplusone.com/weyl

Gavin E. Crooks [2019]
Tech. Note 12v3

CAN$(t_x, t_y, t_z) = \exp\left(-i\frac{\pi}{2}(t_x X \otimes X + t_y Y \otimes Y + t_z Z \otimes Z)\right)$

Instructions:
(1) Print
(2) Cut along outside edges
(3) Fold CPHASE, XY, EXCHANGE, and PSWAP edges
(4) Paste tabs

Source code: https://github.com/gecrooks/weyl

Figure 1: The Weyl chamber of canonical non-local 2 qubit gates. [Print, cut, fold, and paste]
Figure 2: Location of principle 2-qubit gates in the Weyl chamber. All of these gates have coordinates of the form $\text{CAN}(\frac{k}{4}, \frac{m}{4}, \frac{n}{4})$, for integer $k$, $m$, and $n$. 
Table 1: Canonical coordinates of common 2-qubit gates

<table>
<thead>
<tr>
<th>Gate</th>
<th>$t_x$</th>
<th>$t_y$</th>
<th>$t_z$</th>
<th>$t'_x$</th>
<th>$t'_y$</th>
<th>$t'_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CNOT / CZ</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$i$SWAP / DCNOT</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>SWAP</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\sqrt{\text{CNOT}}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{i}$SWAP</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
</tr>
<tr>
<td>DB</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>0</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{\text{SWAP}}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>
2.2 XX gates

Gates in the XX [or Ising] class have coordinates CAN(t, 0, 0), which forms the one edge of the Weyl chamber. This includes the identity and CNOT gates.

**XX [Ising] gate**

\[
XX(t) = e^{-i \frac{\pi}{2} X \otimes X} = \begin{pmatrix}
\cos(\frac{\pi}{2} t) & 0 & 0 & -i \sin(\frac{\pi}{2} t) \\
0 & \cos(\frac{\pi}{2} t) & -i \sin(\frac{\pi}{2} t) & 0 \\
0 & -i \sin(\frac{\pi}{2} t) & \cos(\frac{\pi}{2} t) & 0 \\
-i \sin(\frac{\pi}{2} t) & 0 & 0 & \cos(\frac{\pi}{2} t)
\end{pmatrix}
\]

\[= \text{CAN}(t, 0, 0)\] 

**YY gate**

\[
YY(t) = e^{-i \frac{\pi}{2} Y \otimes Y} \quad \text{(9)}
\]

\[
= \begin{pmatrix}
\cos(\frac{\pi}{2} t) & 0 & 0 & +i \sin(\frac{\pi}{2} t) \\
0 & \cos(\frac{\pi}{2} t) & -i \sin(\frac{\pi}{2} t) & 0 \\
0 & -i \sin(\frac{\pi}{2} t) & \cos(\frac{\pi}{2} t) & 0 \\
+i \sin(\frac{\pi}{2} t) & 0 & 0 & \cos(\frac{\pi}{2} t)
\end{pmatrix}
\]

\[= \text{CAN}(0, t, 0)\]

\[\cong \text{CAN}(t, 0, 0)\]

**ZZ gate**

\[
ZZ(t) = e^{-i \frac{\pi}{2} Z \otimes Z} \quad \text{(10)}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-i \pi t} & 0 & 0 \\
0 & 0 & e^{-i \pi t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[= \text{CAN}(0, 0, t)\]

\[\cong \text{CAN}(t, 0, 0)\]

**Controlled-Y gate**

\[
\text{CY} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & e^{-i \pi t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[\cong \text{CAN}(\frac{1}{2}, 0, 0)\]

Commonly represented by the circuit diagram:

![Controlled-Y gate circuit diagram]

**Controlled-Z gate**

\[
\text{CZ} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & e^{+i \pi t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[\cong \text{CAN}(\frac{1}{2}, 0, 0)\]

Commonly represented by the circuit diagrams

![Controlled-Z gate circuit diagrams]

2.3 XY gates

Gates in the XY class forms two edges of the Weyl chamber with coordinates CAN(t, t, 0) for \(t \leq \frac{1}{2}\) and CAN(t, 1−t, 0) for \(t > \frac{1}{2}\). This includes the identity and iSWAP gates.

**XY-gate** Also occasionally referred to as the piSWAP [or parametric iSWAP] gate.

\[
\text{XY}(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\pi t) & -\sin(\pi t) & 0 \\
0 & \sin(\pi t) & \cos(\pi t) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[= \text{CAN}(t, t, 0)\]

\[\cong \text{CAN}(t, 1−t, 0)\]

**Double Controlled NOT (DCNOT) gate**

\[
\text{DCNOT} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[\cong \text{CAN}(\frac{1}{2}, \frac{1}{2}, 0)\]

**bSWAP (Bell-Rabi) gate**

\[
bSWAP = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix}
\]

\[= \text{CAN}(\frac{1}{2}, \frac{1}{2}, 0)\]

**Dagwood Bumstead (DB) gate** Of all the gates in the XY class, the Dagwood Bumstead-gate makes the biggest sandwiches. [7, Fig. 4]
2.4 Exchange-interaction gates

Includes the identity and SWAP gates.

EXCH (XXX) gate

\[ \text{EXCH}(t) = \text{CAN}(t, t, t) \]  \hspace{1cm} [19]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \cos \left( \frac{3\pi}{8} \right) & \sin \left( \frac{3\pi}{8} \right) \\
0 & -\sin \left( \frac{3\pi}{8} \right) & \cos \left( \frac{3\pi}{8} \right) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \text{DB} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \cos \left( \frac{3\pi}{8} \right) & \sin \left( \frac{3\pi}{8} \right) \\
0 & -\sin \left( \frac{3\pi}{8} \right) & \cos \left( \frac{3\pi}{8} \right) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

= \text{CAN} \left( \frac{3\pi}{8}, \frac{3\pi}{8}, 0 \right)

\[ \text{DB} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \cos \left( \frac{3\pi}{8} \right) & \sin \left( \frac{3\pi}{8} \right) \\
0 & -\sin \left( \frac{3\pi}{8} \right) & \cos \left( \frac{3\pi}{8} \right) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

= \text{XY} \left( \frac{3\pi}{8} \right)

\[ \text{pSWAP gate} \quad [1]
\]

\[ \text{pSWAP}(t) \equiv \text{CAN} \left( \frac{1}{2}, \frac{1}{2}, t \right) \]

\[ \text{2-qubit quantum Fourier transform (QFT)} \quad [1]
\]

\[ \text{QFT} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\]

\[ \equiv \text{CAN} \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \]

2.6 Orthogonal gates

An orthogonal gate, in this context, is a gate that can be represented by an orthogonal matrix (up to local 1-qubit rotations.) The special orthogonal gates have determinant +1 and coordinates \( \text{CAN}(t_x, t_y, 0) \), which covers the bottom surface of the canonical Weyl chamber. The improper orthogonal gates have determinant −1 and coordinates \( \text{CAN}(\frac{1}{2}, t_y, t_z) \), which is a plane connecting the CNOT, iSWAP, and SWAP gates.

\[ \text{pSWAP gates} \]

\[ \text{pSWAP} = \text{CAN} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ \sqrt{\text{SWAP}} = \text{CAN} \left( \frac{3}{4}, \frac{1}{4}, \frac{1}{4} \right) \]

\[ \sqrt{\text{SWAP}} = \text{CAN} \left( \frac{3}{4}, \frac{1}{4}, \frac{1}{4} \right) \]

Special orthogonal gates

Improper orthogonal gates

B (Berkeley) gate \quad [1]

\[ B = \begin{pmatrix}
\cos \left( \frac{\pi}{4} \right) & 0 & 0 & \text{i} \sin \left( \frac{\pi}{4} \right) \\
0 & \cos \left( \frac{\pi}{4} \right) & \text{i} \sin \left( \frac{\pi}{4} \right) & 0 \\
0 & \text{i} \sin \left( \frac{\pi}{4} \right) & \cos \left( \frac{\pi}{4} \right) & 0 \\
\text{i} \sin \left( \frac{\pi}{4} \right) & 0 & 0 & \cos \left( \frac{\pi}{4} \right)
\end{pmatrix}
\]

\[ \equiv \text{CAN} \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \]

Notably two-B gates are enough to create any other 2-qubit gate. Unfortunately the B gate isn’t a particularly natural gate on any current hardware.
\( ECP = \text{CAN}(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \) \hspace{1cm} (25)

2.7 XXY gates

The remaining faces of the Weyl chamber are the XXY family. Thanks to the Weyl symmetries, this family covers all three faces that meet at the SWAP gate.

\[ \text{XXY}(t, \delta) = \text{CAN}(t, t, \delta) \] \hspace{1cm} (26)

2.8 Perfect entanglers

Acknowledgments  Consideration of importance of the canonical gates and gate decompositions arose from many conversations with Eric C. Peterson.

References

[1] [citation needed]. (pages 1, 1, 6, 6, and 6).


